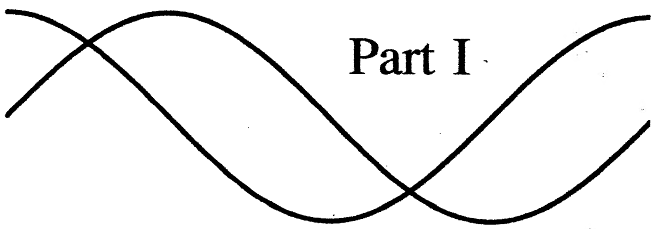
Project MATHEMATICS!

Program Guide and Workbook

to accompany the videotape on

SINES AND COSINES

Part I



Project MATHEMATICS!
CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, CA 91125 • (818) 356-3759

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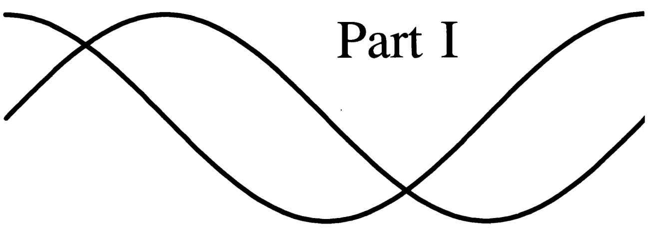

Part I

Written by TOM M. APOSTOL, California Institute of Technology

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SINES AND COSINES I

was produced by Project MATHEMATICS!

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JAMES F. BLINN
JOE CORRIGAN

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AIMS AND GOALS OF *Project MATHEMATICS!*

Project MATHEMATICS! produces computer-animated videotapes to show students that learning mathematics can be exciting and intellectually rewarding. The videotapes treat mathematical concepts in ways that cannot be done at the chalkboard or in a textbook. They provide an audiovisual resource to be used together with textbooks and classroom instruction. Each videotape is accompanied by a workbook designed to help instructors integrate the videotape with traditional classroom activities. Video makes it possible to transmit a large amount of information in a relatively short time. Consequently, it is not expected that all students will understand and absorb all the information in one viewing. The viewer is encouraged to take advantage of video technology that makes it possible to stop the tape and repeat portions as needed.

The manner in which the videotape is used in the classroom will depend on the ability and background of the students and on the extent of teacher involvement. Some students will be able to watch the tape and learn much of the material without the help of an instructor. However, most students cannot learn mathematics by simply watching television any more than they can by simply listening to a classroom lecture or reading a textbook. For them, interaction with a teacher is essential to learning. The videotapes and workbooks are designed to stimulate discussion and encourage such interaction.

STRUCTURE OF THE WORKBOOK

The workbook begins with a brief outline of the video program, followed by suggestions of what the teacher can do before showing the tape. Numbered sections of the workbook correspond to capsule subdivisions in the tape. Each section summarizes the important points in the capsule. Some sections contain exercises that can be used to strengthen understanding. The exercises emphasize key ideas, words and phrases, as well as applications. Some sections suggest projects that students can do for themselves.

I. BRIEF OUTLINE OF THE PROGRAM

The videotape begins with a brief *Review of Prerequisites* dealing with properties of similar figures and the number π, concepts discussed in earlier programs. The student should become familiar with these concepts before viewing the tape. The program opens with examples of circular motion in real life, and then introduces the sine in connection with a point moving counterclockwise on a circle of unit radius. The distance the point moves along the circumference is the radian measure of the corresponding central angle and is recorded on a horizontal t axis. The height y of the moving point above or below the horizontal diameter is called the sine of t, and is written $y = \sin t$. When y is plotted against t the resulting graph is called a sine curve or a sine wave.

By reflecting the sine curve about various lines, some simple properties of the sine are revealed, for example, $\sin(\pi - t) = \sin t$, $\sin(\pi + t) = -\sin t$, and $\sin(-t) = -\sin t$, . Reflection of the sine curve about the line $t = \pi/4$ generates a new curve, called a cosine curve, given by $\cos t = \sin(\pi/2 - t)$. Comparison of graphs reveals that $\cos t = \sin(\pi/2 + t)$. From this it follows that $\cos(-t) = \cos t$.

Next it is shown that a sine wave is generated by recording change in air pressure caused by a vibrating tuning fork. This provides an opportunity to introduce *frequency* (the number of vibrations per second) and *amplitude* (the maximum height of the curve above the axis), and to illustrate these concepts visually and audibly with an electronic synthesizer and with musical instruments.

The periodic nature of the sine curve is emphasized next. Other periodic waves are shown, and the program mentions Fourier's remarkable discovery that every periodic wave is a combination of sine and cosine waves with appropriate amplitudes and frequencies. This is illustrated with sine wave approximations to a square wave.

Next some historical background is given, and it is shown how sines and cosines occur in trigonometry as ratios of lengths of sides of right triangles. Later in the program various properties of sines and cosines are mentioned briefly, for example, the law of cosines, the law of sines, and the addition formulas. These properties are developed in greater detail in a subsequent program entitled *Sines and Cosines, Part II*.

II. BEFORE WATCHING THE VIDEOTAPE

This videotape builds on ideas introduced in two earlier modules, *Similarity* and *The Story of* π. These ideas are listed below and are discussed in the section entitled *Review of Prerequisites*. If students are familiar with these ideas, this section will serve as a review. If not, an effort should be made to acquaint them with these ideas and with the key words and statements listed below before viewing the rest of the tape. A good way to do this is to have the students read the *Review of Prerequisites* section and solve the exercises on page 6.

KEY WORDS AND STATEMENTS:

Expanding or contracting a plane figure by a *scaling factor* produces a *similar* figure, that is, a figure of the same shape but possibly of different size.

A circle of radius r has *circumference* $2\pi r$ and *area* πr^2.

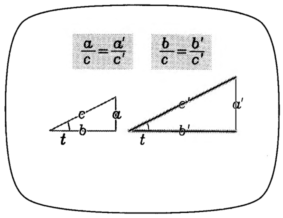
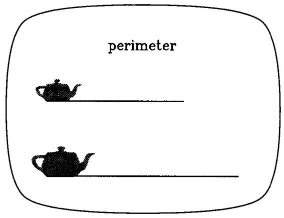
THE MAIN IDEAS:

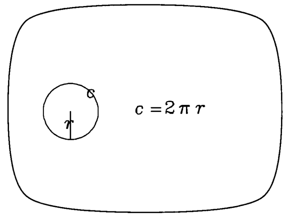
From Similarity: Expanding or contracting a plane figure by a scaling factor multiplies all distances by the same factor. Corresponding angles are equal, lengths of corresponding sides have the same ratio, and corresponding internal ratios are equal. If a figure is scaled by a factor s, lengths of line segments and perimeters are mutliplied by s, surface areas are multiplied by s^2, and volumes are multiplied by s^3.

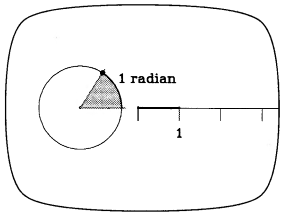
From The Story of π*:* The circumference of a circle of diameter 1 is about 3.14. Scaling by a factor d gives a circle of diameter d and multiplies the circumference by d. The ratio of circumference to diameter is the same for all circles. This ratio, a fundamental constant of nature, is denoted by the Greek letter π. Consequently, the circumference of a circle of diameter d is πd, or, in terms of the radius, $2\pi r$.

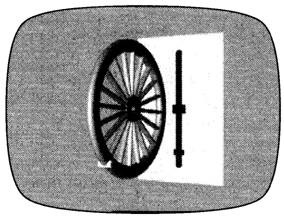
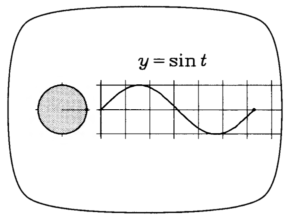
Another ratio that is the same for all circles is the area of a circular disk divided by the square of its radius. Start with a circular disk of radius 1 and denote its area by A. Scaling by a factor r produces a new disk of radius r and area Ar^2. So the ratio Ar^2/r^2 is equal to A, the same constant for all circles. Archimedes (287-212 B.C.), the greatest mathematician of ancient times, made the remarkable discovery that $A = \pi$. This fact is equivalent to the statement that the area of a circular disk of radius r is πr^2. Two different methods of proving that the area is πr^2 are described in another module, *The Story of* π.

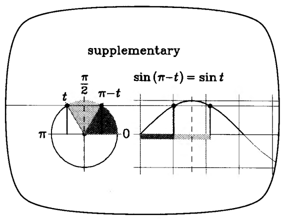
III. TABLE OF CONTENTS

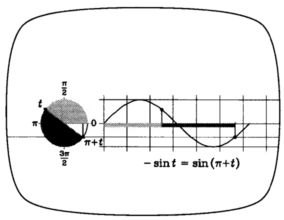



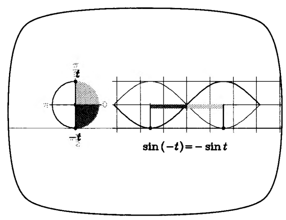


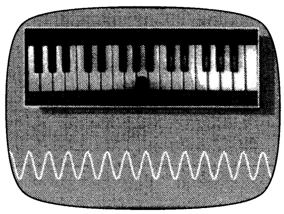
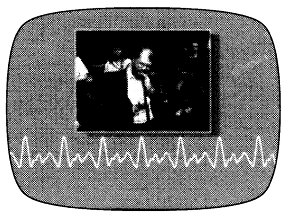
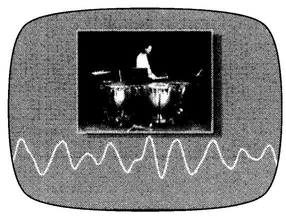


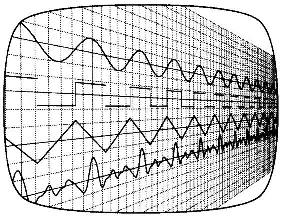



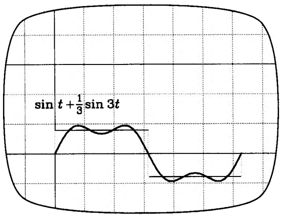


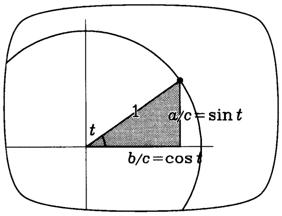


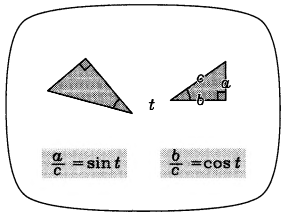


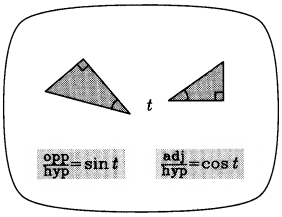




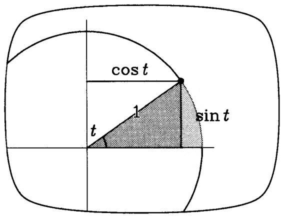


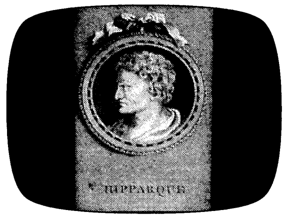

$\sin t + \frac{1}{3}\sin 3t$


t
1
$a/c = \sin t$
$b/c = \cos t$


t
c
a
b
$\frac{a}{c} = \sin t$
$\frac{b}{c} = \cos t$


t
$\frac{\text{opp}}{\text{hyp}} = \sin t$
$\frac{\text{adj}}{\text{hyp}} = \cos t$


$\cos t$
1
$\sin t$
t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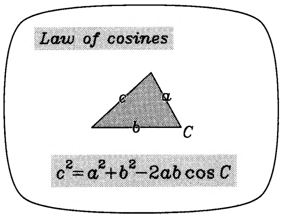
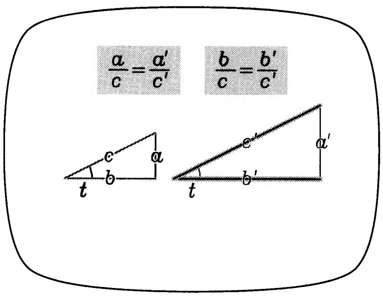
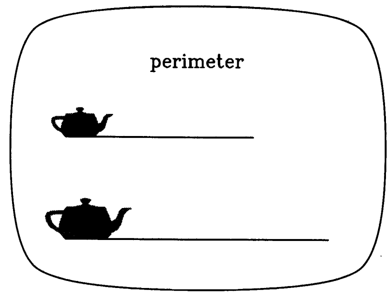
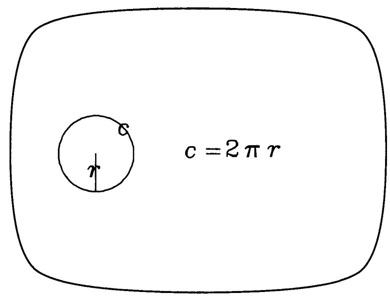

Law of cosines
c
a
b
C
$c^2 = a^2 + b^2 - 2ab\cos C$

Review of prerequisites







Ideas from Similarity: Expanding or contracting a plane figure by a scaling factor multiplies all distances by the same factor. Corresponding angles are equal, lengths of corresponding sides have the same ratio, and corresponding internal ratios are equal.

The perimeter of a triangle is the sum of the lengths of its sides. If each side is multiplied by a factor s, the perimeter is also multiplied by s. The same is true for more general figures. For example, the perimeter of a polygon P is the sum of the lengths of its sides. Scaling by a factor s multiplies the length of each side by s, so the perimeter of the scaled polygon is s times that of P. And the same is true for a curved figure, for example, a circle. The perimeter of a curved figure is the limiting value of perimeters of approximating polygons. This leads to the following general property:

Scaling a plane figure by a factor s multiplies its perimeter by s.

Scaling a triangle by a factor s multiplies the lengths of its sides and its altitudes by s. The area of a triangle is one-half base times altitude. Therefore, when the base and altitude are each multiplied by s, the area is multiplied by s^2. Any polygonal figure can be decomposed into triangular pieces, and its area is the sum of the areas of all the pieces. From this it follows that scaling any polygon by a factor s multiplies its area by s^2. And the same is true for more general plane figures with curved boundaries, because such figures can be approximated from inside and outside by polygons whose areas can be made arbitrarily close to the area of the plane figure. This leads to the following general property:

Scaling a plane figure by a factor s multiplies its area by s^2.

Ideas from The Story of π: We can apply the foregoing properties to circles. Start with a circle of diameter 1 and expand or contract it by a scaling factor d. You get another circle with both the diameter and perimeter multiplied by d. (The perimeter of a circle is also called its *circumference.*) Therefore the ratio of circumference to diameter is the same for all circles, because the scaling factor d cancels when we form the ratio. This constant ratio is denoted by the Greek letter π. The circumference of a circle of diameter d is equal to πd (or $2\pi r$, in terms of the radius r).

If the area of a unit circular disk (of radius 1) is equal to A, then the area of a circular disk of radius r is equal to Ar^2. Archimedes proved that $A = \pi$, so the area of a circular disk of radius r is πr^2.

Because π was recognized as a fundamental constant of nature, people tried for centuries to determine its numerical value accurately. The first serious attempt was made by Archimedes, who obtained approximate values of π by comparing the circumference of a circle with the perimeters of inscribed and circumscribed regular polygons. For a unit circle, an inscribed hexagon can be divided into six equilateral triangles with edge 1. It perimeter, 6, is less than 2π, the circumference of the circle, so $3 < \pi$. Exercise 1 below asks the reader to show that $\pi < 2\sqrt{3}$ by comparing the circle with a circumscribed hexagon.

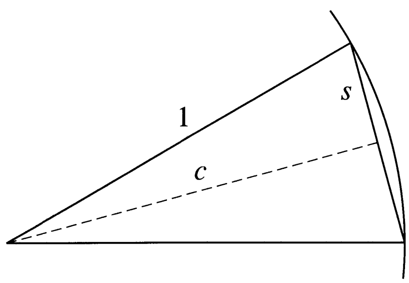
The Story of π explains that π is not a rational number (the ratio of two integers), but it can be approximated to any degree of accuracy by rational numbers. Two rational approximations are 22/7, which gives π correct to two decimals, and 355/113, which gives six decimals of π.

Exercises involving estimates for calculating π

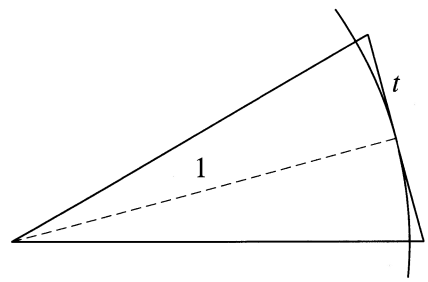
1. Show that a regular hexagon circumscribed about a circle of radius 1 can be decomposed into six equilateral triangles with edge $2\sqrt{3}/3$, and use this to deduce that $\pi < 2\sqrt{3}$.

2. (a) Suppose a regular polygon with n sides is inscribed in a unit circle. The polygon consists of n congruent copies of the isosceles triangle shown in (a). The triangle can be bisected into two right triangles with legs of length s and c, and hypotenuse of length 1, as shown. Prove that the inscribed polygon has perimeter $2ns$ and area ncs.

(b) Suppose a regular polygon with n sides is circumscribed about a unit circle. The polygon consists of n congruent copies of the isosceles triangle shown in (b). The triangle can be bisected into two right triangles with legs of length 1 and t, as shown. Prove that the circumscribed polygon has perimeter $2nt$ and area nt.



(a) Part of the inscribed polygon.



(b) Part of the circumscribed polygon.

(c) Show that the following inequalities for π are obtained by comparing the circumference of the circle with the perimeters of the inscribed and circumscribed regular polygons,

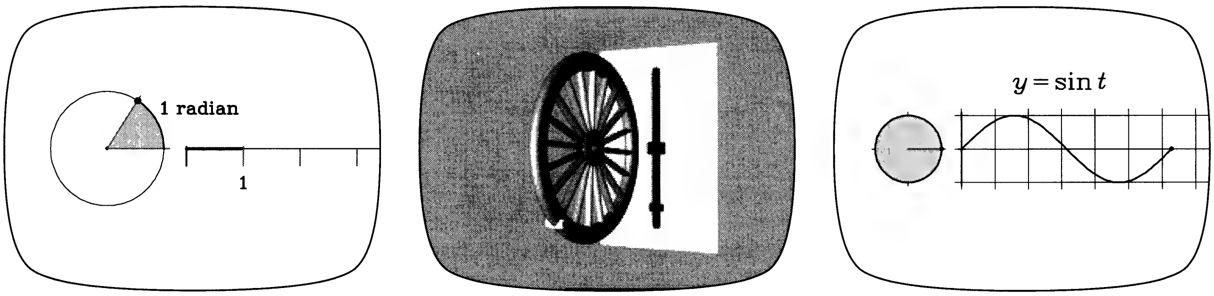
$$ns < \pi < nt,$$

while the following inequalities come from comparison of areas,

$$ncs < \pi < nt.$$

Notice that the same upper bound nt is obtained by using perimeters and areas. But, because $c < 1$, the lower bound ns obtained by using perimeters is larger than ncs obtained by using areas.

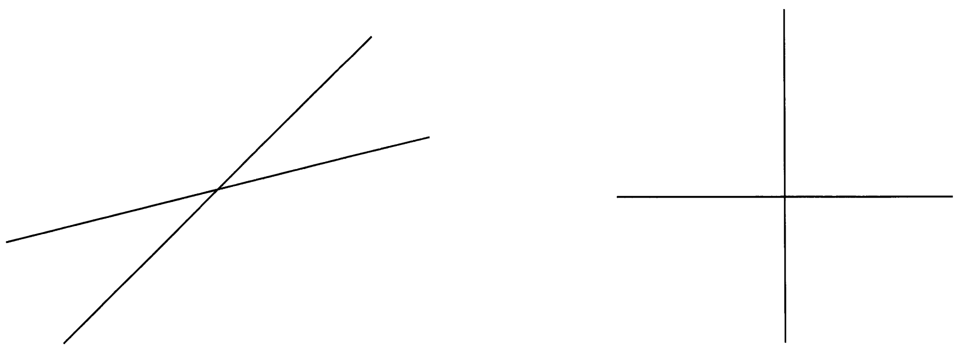
1. *Circular motion and sine waves*



The history of human progress centers around the wheel. The wagon wheel removed the load from man's back and made him better than a beast of burden. The wheel brings water to crops, turns wheat into flour, transports people and goods to the ends of the earth, and converts nature's energy into electricity. Repetitive circular motion is present in machines that dominate our way of life. Circular motion is related to mathematics through the measurement of angles, which we describe next.

Angles and degree measure

Two intersecting lines divide the plane into four regions called *angular sectors*, or *angles*, as shown by the examples in Figure 1. The point of intersection is called the *vertex* of each angle. If the lines are perpendicular, as in Figure 1b, the angles are called *right angles.*



(a) (b)

Figure 1. (a) Angles formed by two intersecting lines. (b) Perpendicular lines form right angles.

We often need to compare the size of angles. The most common unit of measurement is the *degree*, defined to be one ninetieth of a right angle. In other words, a right angle contains ninety degrees, written 90°. Four right angles contain 360°.

Angular measurement can be related to lengths of circular arcs, or areas of circular sectors. With the vertex in Figure 1 as center, draw a circle of convenient radius, as shown in Figure 2. The portions of the angles inside the circle are called *circular sectors.*

Figure 2. Circular sectors used to compare sizes of angles.

When the circumference of the circle is divided into 360 arcs of equal length, each arc is said to subtend an angle of 1°. When the ends of the arcs are joined radially with the center they determine 360 circular sectors, each having area 1/360th that of the circle. An angle of 1° is represented by a very thin circular sector, as shown by the example in Figure 3.

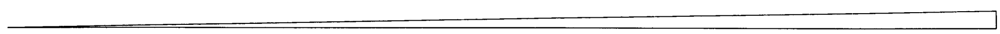


Figure 3. An angle of 1°.

Degree measure has been used since antiquity and is widely accepted, although there is no *a priori* reason for dividing a right angle into ninety equal parts. Perhaps early mathematicians and astronomers wanted 360° in a complete circle because 360 is divisible by many integers. In fact, its divisors are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, and 360. Therefore there are many ways to divide a circle into equal parts with each part containing a whole number of degrees. To avoid using fractions of a degree, the degree itself is subdivided into sixty equal parts called *minutes,* and the minute into sixty equal parts called *seconds.* An angle of 7 degrees, 12 minutes and 5 seconds is written $7^{\circ}\ 12'\ 5''$. In this context, the words *minute* and *second* have nothing to do with time. They originate from Latin terms used by scholars translating Ptolemy's division of the degree into *partes minutae primae* and each of these minutes into *partes minutae secundae.*

Angles can be measured with an instrument called a protractor, shown in Figure 4. Most protractors have two scales so you can measure an angle in either the clockwise or counterclockwise direction.

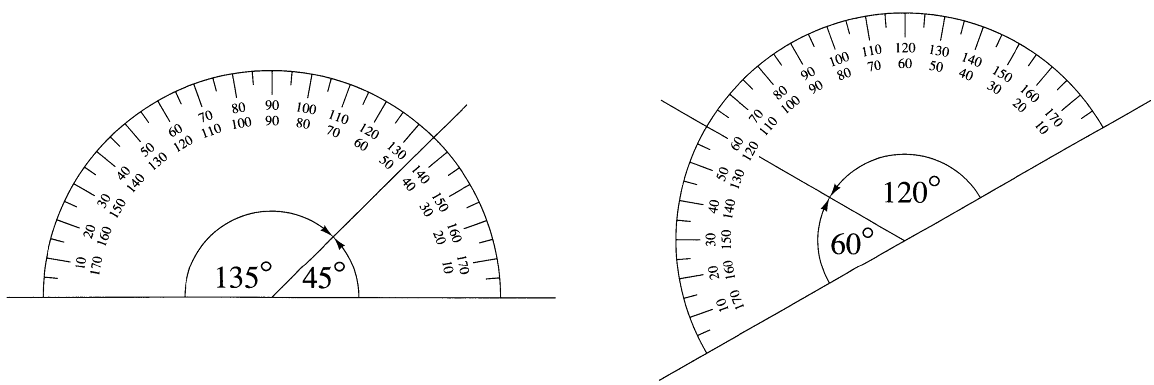


Figure 4. A protractor used to measure the number of degrees in an angle.

Degree measure is commonly used in engineering and astronomical applications for measuring angles of triangles or arcs of circles. But there is another common unit of angular measure called the *radian* that is more natural than the degree and is widely used in mathematics.

Radian measure of angles using arc length

The radian is defined as that angle which, when placed at the center of a circle, subtends an arc equal in length to the radius, as shown in Figure 5a. One radian contains slightly more than 57°, so it is a much larger unit than the degree. Radian measure does not have the artificial quality of degree measure. It is more natural to divide the circumference of a circle into units equal to the length of the radius than to divide it into 360 equal parts.

To find the radian measure of an arbitrary central angle, such as that shown in Figure 5b, simply take the ratio of the arc length s to the radius r. The ratio s/r is called the *radian measure* of the angle. For example, a circle of radius r contains 2π radians (the ratio of circumference to radius is $2\pi r/r = 2\pi$). If $s/r = \theta$ the measure of the angle is θ (θ is the Greek letter theta, often used to denote angular measure).

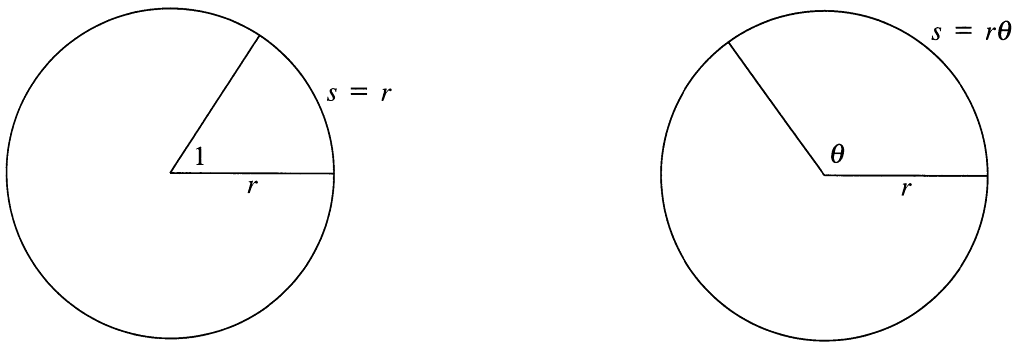
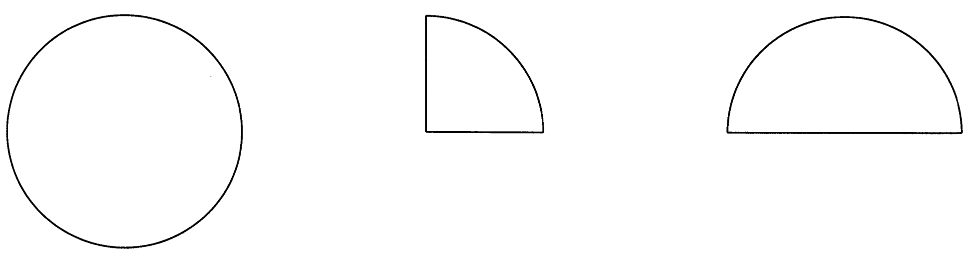


Figure 5. (a) An angle of one radian. (b) An angle of θ radians.

Because s/r is the ratio of two distances, it is ''dimensionless,'' that is, it is a number independent of the unit of distance that is being used to make measurements. (Degree measure is also dimensionless.) If the radian angular measure θ is known, the arc length s can be determined from the formula $s = r\theta$. This simple equation for the length of a circular arc is valid only if θ is measured in radians.

There is a simple relation between degree measure and radian measure. For example, a circle contains 360° or 2π radians, a right angle contains 90° or $\pi/2$ radians, and a straight angle (two right angles) contains 180° or π radians (see Figure 6).



2π radians $= 360^{\circ}$ $\frac{\pi}{2}$ radians $= 90^{\circ}$ π radians $= 180^{\circ}$

Figure 6. Comparison of radian measure with degree measure.

More generally, to convert an angle of α degrees to θ radians we note that the two ratios $\alpha/360$ and $\theta/(2\pi)$ represent the same fraction of a circle, so

$$\frac{\theta}{2\pi} = \frac{\alpha}{360} \quad \text{or} \quad \theta = \frac{\pi}{180}\alpha.$$

The word *radian* is a contraction of *radial angle*, and was introduced in the late nineteenth century by mathematician Thomas Muir and physicist James T. Thomson. It first appeared in print in an examination paper set by Thomson in 1873.

Radian measure of angles using area

Another way to obtain radian measure is to use the area of the sector instead of the arc length, as shown in Figure 7. The area, a, bears the same relation to the total area of the circular disk, πr^2, that the arc length s bears to the total circumference, $2\pi r$. That is,

$$\frac{a}{\pi r^2} = \frac{s}{2\pi r},$$

which implies

$$\frac{s}{r} = \frac{2a}{r^2}.$$

But $s/r = \theta$, the radian measure of the angle, and hence $\theta = 2a/r^2$. In other words, the radian measure θ is twice the area of the sector divided by the square of the radius.

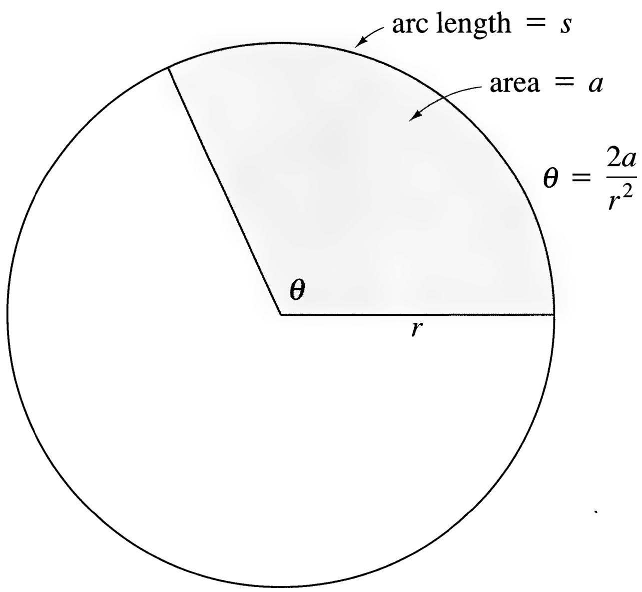


Figure 7. Radian measure expressed in terms of the area of the sector.

Exercises on angular measure

1. Express the following in radian measure in terms of π: (a) 30° (b) 45° (c) 60° (d) 135° (e) 150°

2. Express each of the following radian measures in degrees: (a) $\pi/6$ (b) $\pi/5$ (c) $\pi/4$ (d) $\pi/3$ (e) $5\pi/6$

3. The French once proposed dividing a right angle into 100 equal units, called *grades*. Determine the number of grades in each of the following:

(a) 1 degree (b) 1 radian (c) $\pi/2$ radians (d) 45°

(e) A circular sector whose arc is equal in length to the diameter of the circle.

4. Show that the area of a circular sector of radius r subtending an angle of θ radians is equal to $\theta r^2/2$.

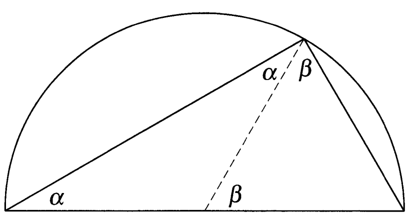
5. Show that the area of a circular sector of radius r subtending an arc of length s is equal to $sr/2$. This formula is of interest because it suggests that for calculating areas, a circular sector can be treated as if it were a triangle with base s and altitude r.

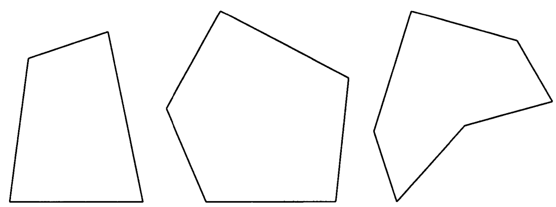
6. Because a rectangle can be decomposed into two right triangles, the sum of the angles in a right triangle is 180°, or π radians. Use this fact to prove that the sum of the angles in *any* triangle is 180°.

Note. In this workbook we have not made a distinction between an *angle* and *the measure of an angle*. Strictly speaking, they are conceptually distinct. An angle is a geometric object, while its measure is a number that records its size. However, it is common practice (and no harm is done) to say "the sum of the angles is 180°" rather than "the sum of the measures of the angles is 180°." Similarly, we often make statements such as "two angles are equal" rather than "two angles have equal measure."

7. Use Exercise 6 to show that any triangle inscribed in a semicircle with one side along a diameter is a right triangle with the diameter as hypotenuse. *Hint:* Determine the sum $\alpha + \beta$ in the diagram below.

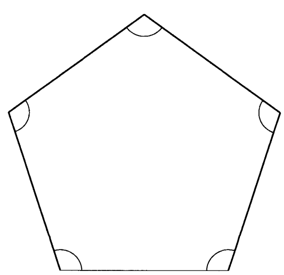
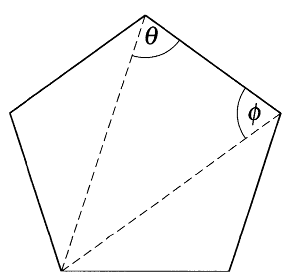
8. Determine the sum of the angles in each of the three polygons shown.





Exercise 7. Exercise 8.

9. Each of the following is a regular pentagon (all edges have equal length). Prove that in (a) all the angles are equal, and in (b) $\theta = \phi = 72^\circ$.

(a)



(b)

The sine and cosine functions defined on the interval from 0 *to* 2π

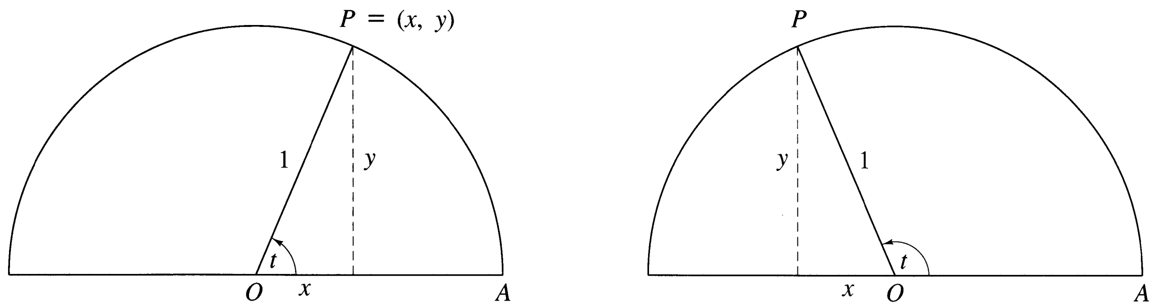
When you look at a bright spot moving around a circle, what you actually see depends on your point of view. You see a true circle only if you view it from a direction perpendicular to the plane of the circle. As the plane rotates about a vertical diameter, the spot appears to move along an oval curve, called an ellipse. And when the plane has rotated 90°, the motion appears straight up and down, as though the spot was casting a shadow. The height of the spot above or below a horizontal diameter is of special interest.

Take a circle of radius 1, called a *unit circle*, and place its center at the origin O of a rectangular coordinate system. The circumference of the circle is equal to 2π. The circle intersects the horizontal x axis at the points $(1, 0)$ and $(-1, 0)$ and the vertical y axis at $(0, 1)$ and $(0, -1)$.

Now suppose a point P moves counterclockwise on the circle, beginning at the point $A = (1, 0)$. The location of P on the circle can be described by the angle that the radial line OP makes with the positive x axis, as shown in Figure 8. Let t denote the radian measure of this angle, and let x and y denote the rectangular coordinates of P. The coordinates can be positive, negative, or zero. Each coordinate is completely determined by t and is, therefore, a function of t. The horizontal coordinate x is called the *cosine* of t, and the vertical coordinate y is called the *sine* of t, and we write

$$x = \cos t, \qquad y = \sin t.$$

Because of their relation to the unit circle the sine and cosine functions are called *circular functions.*



(a) $0 < t < \pi/2$; both x and y are positive. (b) $\pi/2 < t < \pi$; x is negative, y is positive.

Figure 8. The horizontal x coordinate of P is $\cos t$, the vertical y coordinate is $\sin t$.

If $0 < t < \pi/2$, as in Figure 8a, both coordinates x and y are positive. If t is greater than $\pi/2$ but less than π, as in Figure 8b, the first coordinate $\cos t$ is negative, but the second coordinate $\sin t$ is positive.

If $\pi < t < 3\pi/2$, both coordinates $\cos t$ and $\sin t$ are negative, and if $3\pi/2 < t < 2\pi$, the first coordinate $\cos t$ is positive, and second coordinate $\sin t$ is negative.

The large diagram in Figure 9 can be used to estimate the values of $\cos t$ and $\sin t$ correct to two decimals for various values of t measured in increments of 0.1 radians.

Note. The word *sine* is the anglicized form of the Latin word *sinus*, introduced around 1150 A.D. in a Latin translation of an Arabic manuscript containing tables of chords of circles. The word *cosine* is short for *complementary sine*, and is due to Edmund Gunter (1620) who suggested combining the terms *complement* and *sinus* into *co.sinus,* which was soon modified and anglicized to *cosine*.

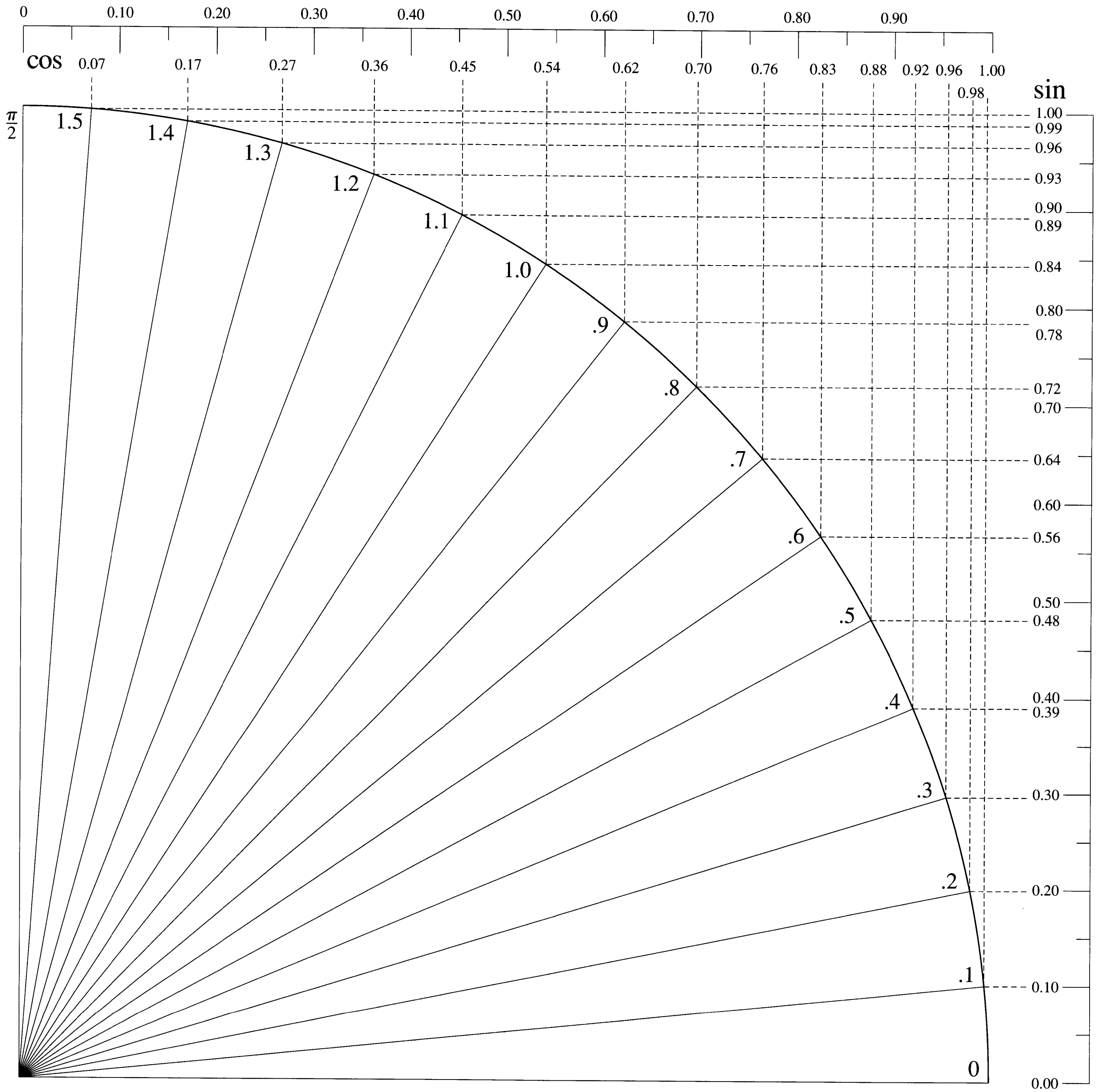


Figure 9. A diagram for estimating cosines and sines of angles measured in increments of 0.1 radians. The circle has unit radius. The horizontal scale indicates the cosine, the vertical scale the sine.

By careful measurement of the coordinates of P, we can draw graphs of the sine and cosine of an angle as the angle varies over the interval from 0 to 2π. These graphs are shown in Figure 10, where the variable angle is denoted by x.

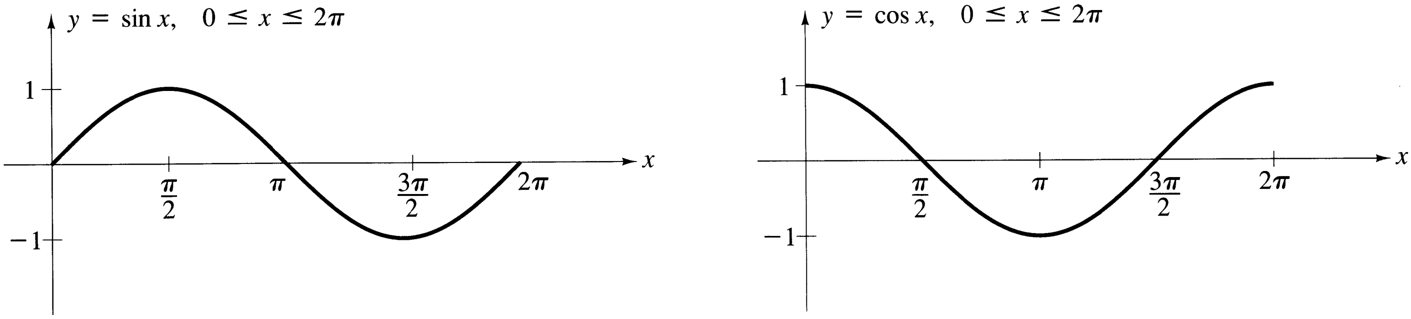


Figure 10. Graphs of the sine and cosine functions over the interval $[0, 2\pi]$.

The values of the sine and cosine lie between -1 and $+1$. We note, in particular, the following values:

$$\cos 0 = 1, \qquad \cos \frac{\pi}{2} = 0, \qquad \cos \pi = -1, \qquad \cos \frac{3\pi}{2} = 0,$$
$$\sin 0 = 0, \qquad \sin \frac{\pi}{2} = 1, \qquad \sin \pi = 0, \qquad \sin \frac{3\pi}{2} = -1.$$

Exercises on determining special values of the sine and cosine

1. Verify the following entries in a table of sines by dividing a unit circle with center at the origin into eight equal parts and determining the rectangular coordinates of the subdivision points. The angle t is measured in radians.

t	$\pi/4$		$3\pi/4$	$5\pi/4$	$7\pi/4$
$\sin t$	$\sqrt{2}/2$	$\sqrt{2}/2$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	

2. Fill in the values of $\cos t$ in the following table.

t	$\pi/4$		$3\pi/4$	$5\pi/4$	$7\pi/4$
$\cos t$					

3. Verify the following entries in a table of cosines by dividing a unit circle with center at the origin into twelve equal parts and determining the rectangular coordinates of the subdivision points.

t	$\pi/6$		$\pi/3$	$2\pi/3$	$5\pi/6$	$7\pi/6$	$4\pi/3$	$5\pi/3$	$11\pi/6$
$\cos t$	$\sqrt{3}/2$	$1/2$	$-1/2$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$-1/2$	$1/2$	$\sqrt{3}/2$	

4. Fill in the values of sin t in the following table.

t	$\pi/6$	$\pi/3$	$2\pi/3$	$5\pi/6$	$7\pi/6$	$4\pi/3$	$5\pi/3$	$11\pi/6$
$\sin t$								

Extending the definitions of the sin x and cos x for all real values of x

The foregoing discussion defines $\sin x$ and $\cos x$ as functions of x for all x in the interval $[0, 2\pi]$. We now extend the definitions of the sine and cosine to all real values of x by making them *periodic functions* with *period* 2π. This means that

$$\sin(x + 2\pi) = \sin x \quad \text{and} \quad \cos(x + 2\pi) = \cos x$$

for all real x. Geometrically, the graphs of the sine and cosine over the interval $[0, 2\pi]$ are repeated in successive intervals of length 2π, as shown in Figure 11.

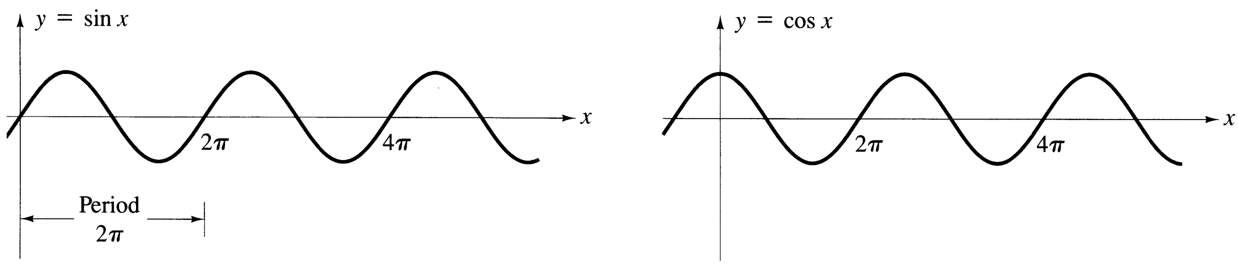


Figure 11. Periodic extension of the sine and cosine functions.

Every real number x can be interpreted as the radian measure of some angle. If x is between 0 and 2π, we regard x as the measure of the angle swept out by the point P in Figure 8 moving counterclockwise along the unit circle. If x is greater than 2π we think of the angle being swept out by a point P that has moved counterclockwise more than once around the circle. And if x is negative, we think of the angle as being swept out by a point P moving in the opposite, or clockwise, direction. Therefore, the definitions of $\cos x$ and $\sin x$ as the horizontal and vertical coordinates of P are meaningful for all real values of x. In many applications the number x often represents time, or distance, or some other quantity that may be of interest. This interpretation makes it possible to employ the sine and cosine functions in the study of general periodic phenomena.

Thus the sine and cosine are examples of *periodic* functions. In general, a function $f(x)$ is said to have period p if

$$f(x + p) = f(x) \quad \text{for all real } x.$$

This means that the value of the function is unchanged when x is replaced by $x + p$. The circular functions $\cos x$ and $\sin x$ have period 2π.

Exercises on periodic functions

1. Show that $\sin(x + 2n\pi) = \sin x$ and $\cos(x + 2n\pi) = \cos x$ for every integer n.

2. Use Exercise 1 to determine the values of $\sin x$ and $\cos x$ for each of the following values of x:

$$3\pi, \quad 7\pi/3, \quad 9\pi/2, \quad -5\pi, \quad -25\pi/6, \quad 13\pi/4, \quad -10\pi/3 .$$

3. If a function $f(x)$ has period p show that it also has period $2p$, $3p$, and more generally np, for any positive or negative integer n.

4. If each of two functions $f(x)$ and $g(x)$ has period p, show that $Af(x) + Bg(x)$ has period p for every choice of constants A and B.

5. Find a nonzero period for each of the following functions:

   (a) $\sin 2x$.

   (b) $\sin 2\pi x$.

   (c) $\sin x + \sin 2x$.

6. (a) Prove that $\sin n\pi = 0$ for every integer n, and then show that these are the only values of x for which $\sin x = 0$.

   (b) Find all real x such that $\cos x = 0$.

7. Prove that the functions $\sin x$ and $\cos x$ have no positive period p smaller than 2π.

8. Find a positive period $p < 2\pi$ for each of the following functions:

   (a) $(\sin x)/(\cos x)$, defined when $\cos x$ is not zero.

   (b) $\cos^2 x$.

   (c) $\sin^2 x$.

   (d) $\cos^2 x - \sin^2 x$.

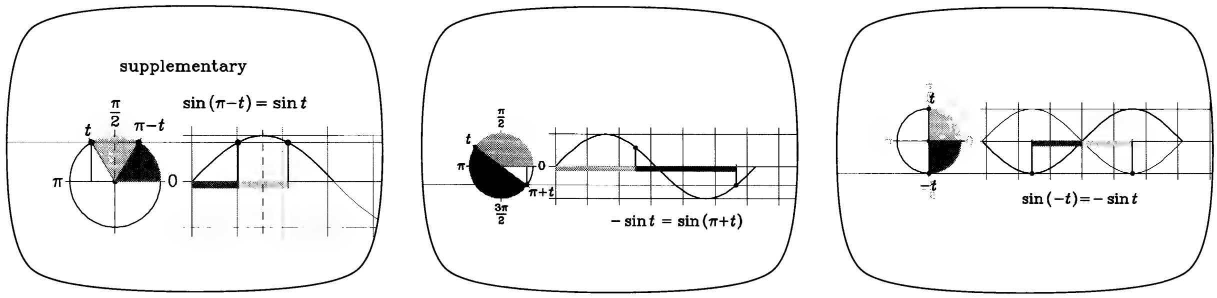
   (e) $\cos^2 x + \sin^2 x$.

9. Find the smallest positive constant k such that the function $f(x) = \sin kx$ has

   (a) period 5.

   (b) period p, where p is positive.

2. *Symmetry of sine waves*



Symmetry properties of the circle give rise to symmetry properties of sine curves. For example, as the angle t increases from 0 to $\pi/2$ radians the function $\sin t$ increases from 0 to 1. As the angle continues to increase from a right angle to a straight angle (π radians) the sine curve drops back to zero, forming a symmetric arch as shown in Figure 12. The arch is symmetric about the line $t = \pi/2$ because the circle is symmetric about its vertical diameter. Two angles t and $\pi - t$ whose sum is π are called *supplementary angles*. The symmetry of the arch implies that supplementary angles have the same sine; that is, the sine of an angle is equal to the sine of its supplement:

$$\sin t = \sin(\pi - t).$$

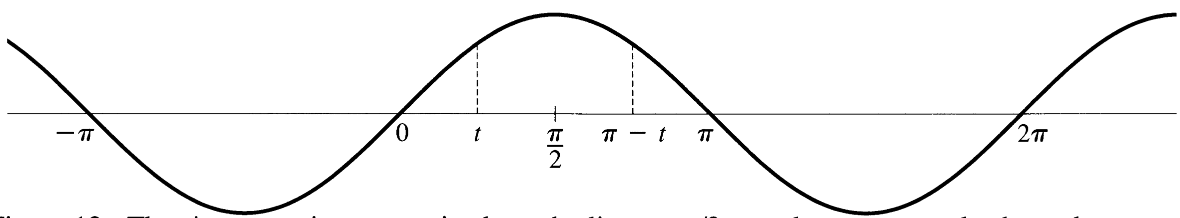


Figure 12. The sine curve is symmetric about the line $t = \pi/2$: supplementary angles have the same sine.

Because the circle is also symmetric about its horizontal diameter, when t varies from π to 2π the sine is negative and its graph has the same shape as the first arch, except it is flipped over. The sine of the angle $\pi + t$ is the negative of the sine of t:

$$\sin(\pi + t) = - \sin t.$$

Note also that $\sin(-t) = - \sin t$. This is called the *odd property* of the sine. These properties are illustrated in Figure 13.

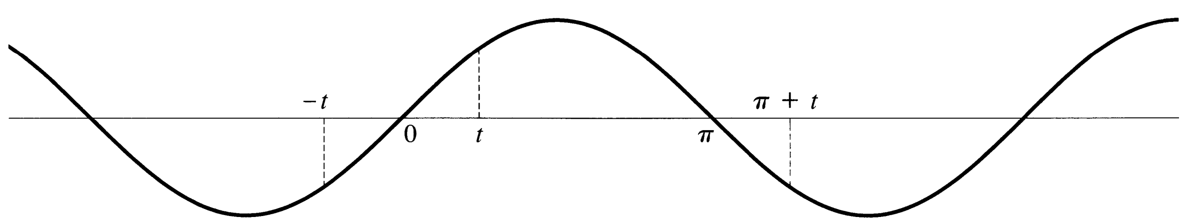


Figure 13. Geometric meaning of the properties $\sin(\pi + t) = - \sin t$ and $\sin(-t) = - \sin t$.

Relations between sines and cosines

Figure 14 shows a cosine curve and a sine curve. The diagram reveals that the two curves have the same shape but the cosine curve is shifted to the left by $\pi/2$ radians. This means that the cosine of t is equal to the sine of $t + \pi/2$:

$$\cos t = \sin(\tfrac{\pi}{2} + t).$$

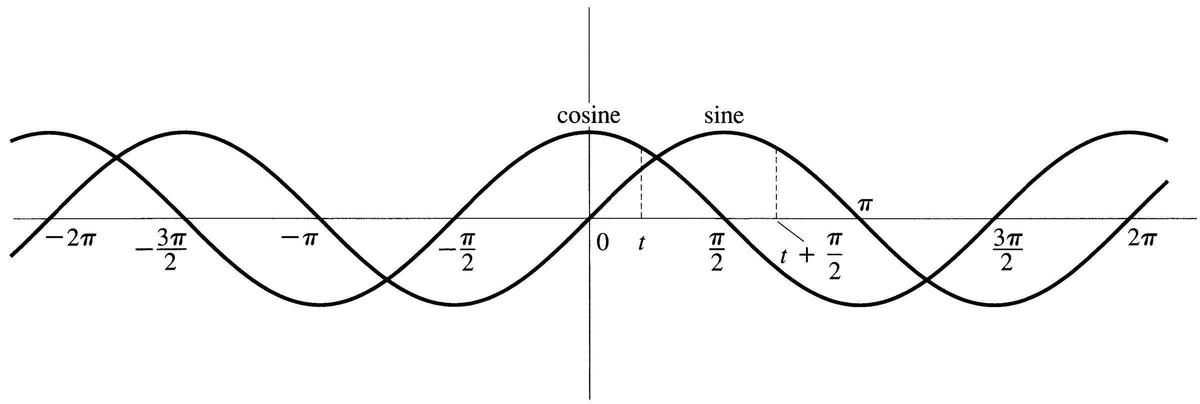


Figure 14. The cosine curve is a sine curve shifted to the left by $\pi/2$ radians.

The angles $\frac{\pi}{2} + t$ and $\frac{\pi}{2} - t$ are supplementary (their sum is π) so they have the same sine. In symbols,

$$\sin(\tfrac{\pi}{2} + t) = \sin(\tfrac{\pi}{2} - t).$$

Using this in the last displayed equation we obtain

$$\cos t = \sin(\tfrac{\pi}{2} - t).$$

Replacing t by $\frac{\pi}{2} - t$ in this equation we find

$$\cos(\tfrac{\pi}{2} - t) = \sin t.$$

Two angles t and $\frac{\pi}{2} - t$ whose sum is $\frac{\pi}{2}$ are called *complementary angles*. The last two equations tell us that the cosine of an angle is the sine of its complement, and the sine of an angle is the cosine of its complement.

Note also that $\cos(-t) = \sin\{\frac{\pi}{2} - (-t)\} = \sin(\frac{\pi}{2} + t) = \cos t$. The equation

$$\cos(-t) = \cos t$$

is called the *even property* of the cosine.

Complementary angles occur in right triangles. Figure 15a shows a right triangle OAP with a right angle at vertex A. The perpendicular sides OA and AP are called legs of the triangle, and the third side OP is called the hypotenuse. Denote the measure of the angle at vertex O by θ and that at vertex P by ϕ. Because the sides OA and AP are perpendicular, two copies of the triangle form a rectangle, as shown

in Figure 15b. Each corner of the rectangle is a right angle, so $\theta + \phi = 90^o$ if the angles are measured in degrees, or $\theta + \phi = \pi/2$ if the angles are measured in radians. Therefore the angles θ and ϕ are complementary angles, and hence we have the relations

$$\sin\theta = \cos\phi \text{ and } \cos\theta = \sin\phi.$$

Figure 15. Complementary angles θ and ϕ in a right triangle: $\sin\theta = \cos\phi$ and $\cos\theta = \sin\phi$.

In Section 5 we will show that the sine and cosine represent ratios of lengths of sides of right triangles.

Exercises relating sines and cosines

Use the relations between sines and cosines already established to deduce the formulas in Exercises 1 through 6. Use the graphs of the sine and cosine to interpret each formula geometrically.

1. $\cos(t + \frac{\pi}{2}) = -\sin t.$

2. $\cos(\pi - t) = -\cos t.$

3. $\cos(\pi + t) = -\cos t.$

4. $\sin(\frac{\pi}{4} + t) = \cos(\frac{\pi}{4} - t).$

5. $\cos(\frac{3\pi}{2} + t) = \sin t.$

6. $\sin(\frac{3\pi}{2} + t) = -\cos t.$

In each of Exercises 7 through 10, determine all values of t in the interval $[0, 2\pi]$ (if any exist) that satisfy the given equation.

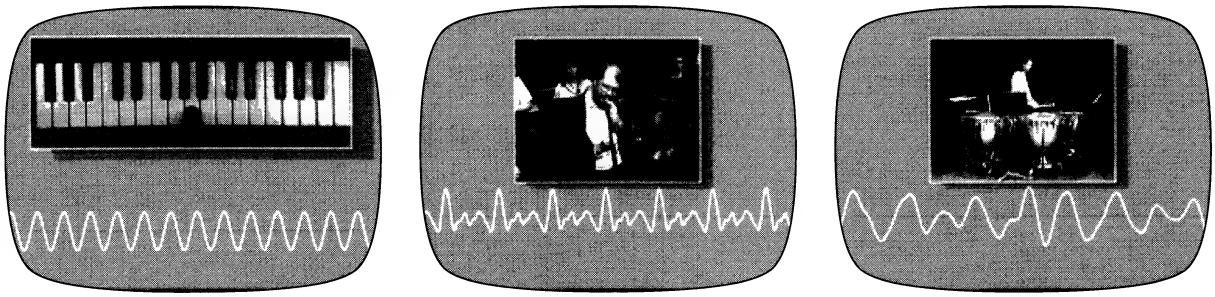
7. $\sin t = \cos t.$

8. $\sin t = -\cos t.$

9. $\sin t = \sin(t + \frac{\pi}{2}).$

10. $\cos t = \cos(2\pi - t).$

3. *Sine waves and sound*



Sine waves also occur in many vibrating phenomena that have nothing to do with circles. For example, Figure 16 shows a vibrating tuning fork that produces a musical tone, or sound wave. Sound waves reach our ears as fluctuations in air pressure. The vibrating tuning fork sets the air next to it in motion, alternately compressing then expanding it with each vibration. Each layer of air passes the motion on to the next layer. The variation of sound pressure can be detected with a microphone connected to a cathode-ray oscilloscope that traces out a curve showing how the sound pressure varies with time. The curve is called the waveform of the sound wave.

When a tuning fork is struck gently it generates a pure tone in which the air pressure rises and falls sinusoidally with time; the waveform has the shape of a sine curve.

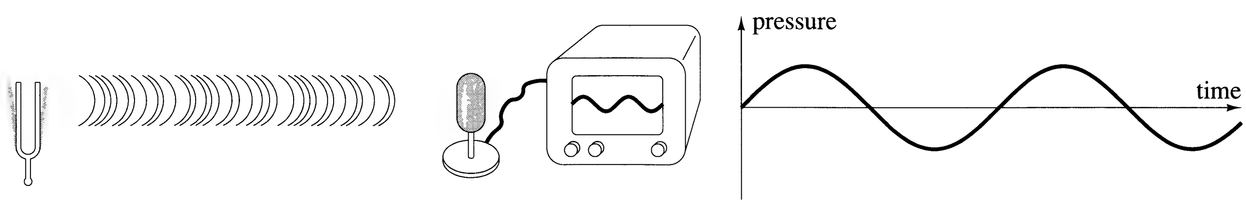


Figure 16. A vibrating tuning fork generates a waveform resembling a sine wave.

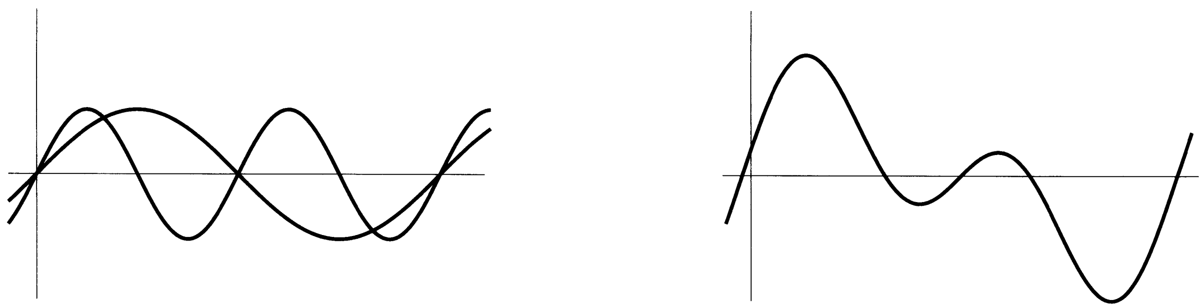
The number of vibrations per second is called the *frequency, f*. It depends on the pitch of the tuning fork. For example, if the fork produces 440 pulses per second we hear the tone A above middle C. If there are 220 pulses per second, the sound is an octave lower, and if there are 880 the sound is an octave higher, called an *overtone*. The height of the curve, up or down, depends on the loudness or intensity of the sound. The maximum height of the curve above the axis is called its *amplitude*. The *period, T*, of a sine wave is the time between amplitude peaks, measured in seconds. The reciprocal of T is therefore the number of peaks per second, and this is the frequency f of the sine wave. Period and frequency are related by the equation

$$f = 1/T.$$

The waveform of a combination of two tones can be obtained by adding the y coordinates of the waveforms of the individual tones. Figure 17a shows two waveforms, one representing a pure tone $y = \sin t$ and another representing an overtone $y = \sin 2t$. The waveform of the combination is described by the equation

$$y = \sin t + \sin 2t,$$

whose graph shown in Figure 17b.



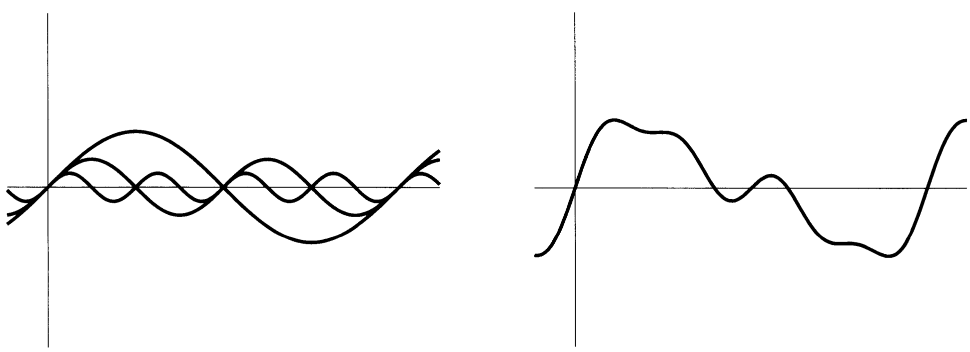
(a) Tone $y = \sin t$ and overtone $y = \sin 2t$ (b) Waveform of the combination: $y = \sin t + \sin 2t$

Figure 17. The waveform of a combination of two tones is obtained by adding the individual waveforms.

When a particular note is played on different instruments, such as a flute, a clarinet, and a french horn, the sound emitted will have a different quality because each instrument produces a different combination of overtones and amplitudes. For example, Figure 18a shows the waveforms of a tone $y = \sin t$, and two overtones with different amplitudes,

$$y = \frac{1}{2} \sin 2t, \qquad y = \frac{1}{4} \sin 4t.$$

When these three waves are added together the result is the waveform shown in Figure 18b.

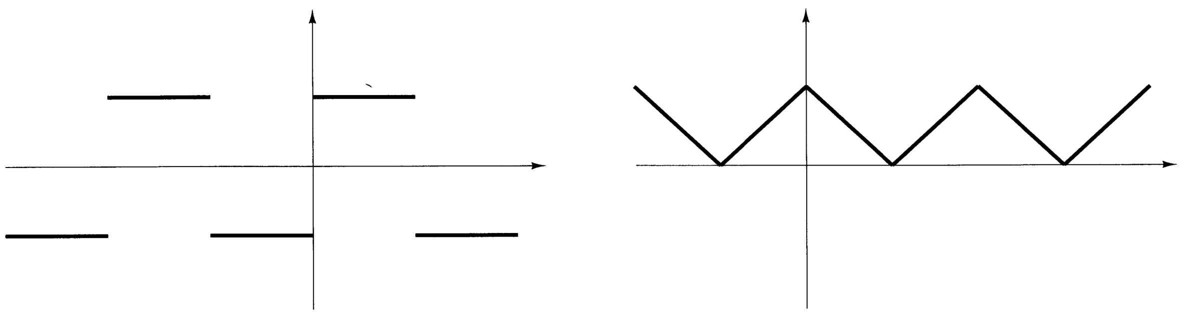


(a) (b)

Figure 18. (a) Three waveforms with different frequencies and amplitudes, and (b) their sum.

4. *Periodic waves*

A sine wave is periodic because it consists of a basic shape repeated indefinitely. Other examples of periodic waves include a *square wave* (Figure 19a), and a *saw-tooth wave* (Figure 19b). To construct a periodic wave of your choice, select an interval to represent the period, draw a graph that has equal y coordinates at the endpoints of the interval and repeat the graph indefinitely, as suggested by the examples in Figure 19.



(a) A square wave. (b) A saw-tooth wave.

Figure 19. Examples of periodic waves.

In the early 19th century the French mathematician Joseph Fourier (1768-1830) astounded the mathematical world when he announced that *every* periodic wave is a combination of sine and cosine waves with appropriate amplitudes and frequencies. Moreover, the frequencies of the component waves are related in a simple way: each frequency is an integer multiple of a single frequency.

Some periodic waves may require a large number of sines and cosines to represent them, even an infinite number. For example, the simple-looking square wave in Figure 19a of frequency f can be represented as a sum of infinitely many sine waves having frequencies that are odd multiples of f: f, $3f$, $5f$, $7f$, and so on, with corresponding amplitudes 1, 1/3, 1/5, 1/7,... . The sum of a finite number of these waves gives an approximation to the square wave, and the approximation gets better and better as more terms are used. Figure 20 shows an approximation using a combination of five sine waves.

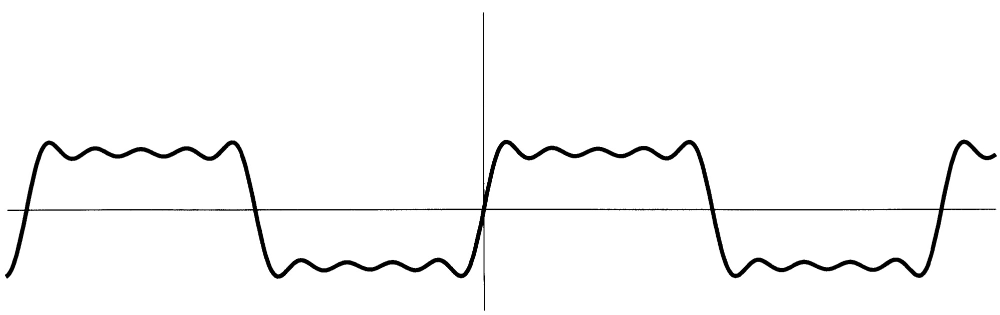


Figure 20. A square wave approximated by a combination of five sine waves.

The saw-tooth wave in Figure 19b can be represented by an infinite number of cosine waves with odd multiple frequencies f, $3f$, $5f$, $7f$,..., and with corresponding amplitudes 1, $(1/3)^2$, $(1/5)^2$, $(1/7)^2$, ... Figure 21 shows a combination of five cosine waves that approximates the saw-tooth wave.

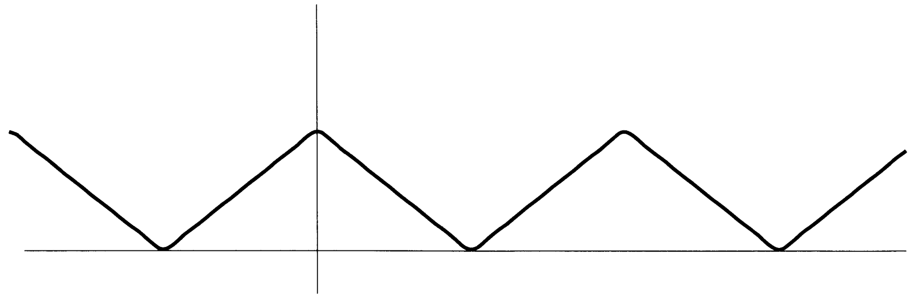


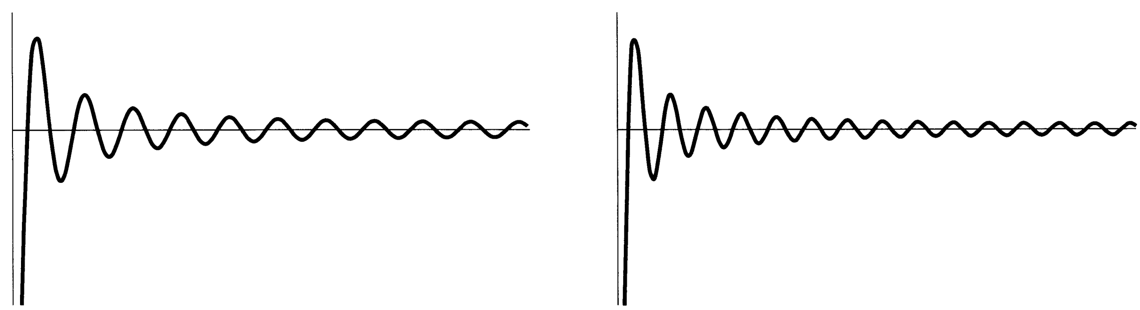
Figure 21. A saw-tooth wave approximated by a combination of five cosine waves.

The examples in Figures 20 and 21 are not typical because they are made up of only odd frequency components. Some wind instruments and closed organ pipes produce sounds that have predominantly odd frequency components. However, most periodic waves contain both odd and even frequency components.

A pure tone produced by a musical instrument is only *part* of a periodic wave because the tone starts, persists for a while, then dies away. A complete periodic wave would have to persist from the infinite past to the infinite future. Representing musical sounds by combinations of sine and cosine waves is an idealized mathematical model of reality. Nevertheless, this model is quite useful and, in fact, is the basis for producing electronic music. A device called a synthesizer, controlled by an electronic keyboard, produces music by combining basic sine and cosine waves. A piece of music produced on instruments can be analyzed and decomposed into their Fourier sine and cosine components. The synthesizer can recombine the sine and cosine waves to reproduce a facsimile of the original piece of music.

The Gibbs phenomenon

Figure 22 shows close-up views of two combinations of sine waves approximating a square wave. Note that the sine waves oscillate closer and closer to the square wave everywhere except near the points where the square wave has a discontinuity. For example, near $x = 0$ the highest maximum, or peak, of the approximating curve moves left as more and more sine waves are combined, but the height of the peak remains above the square wave by about 16%. This overshooting is called the *Gibbs phenomenon*, in honor of Josiah Willard Gibbs, a Yale physicist, who observed it in 1898. It was first observed 50 years earlier by Henry H. Wilbraham, a mathematician from Cambridge University.

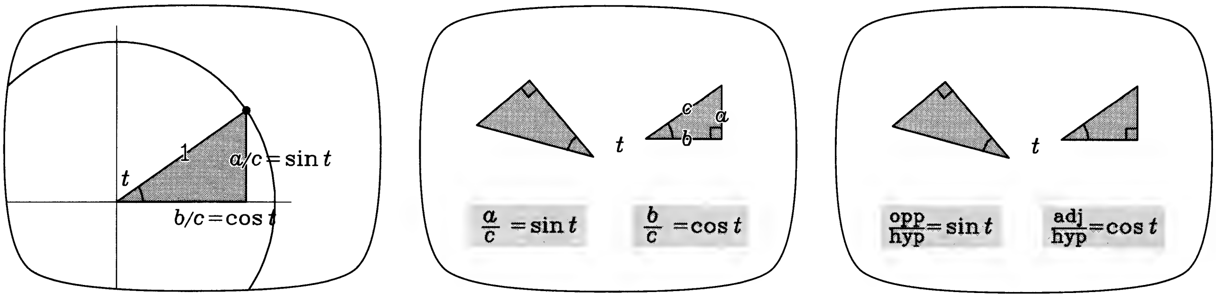


Combination of 20 sine waves

Combination of 30 sine waves

Figure 22. The Gibbs phenomenon. The highest peak remains above the square wave by about 16%.

4. *Sines and cosines as ratios*



Sines and cosines originated in early astronomy. Because planets were thought to move in circular orbits, early astronomers were interested in chords of circles. Babylonian clay tablets have been found containing lists of chord lengths used for astronomical data.

Around 140 B.C. the Greek mathematician Hipparchus, who is regarded as the father of trigonometry, wrote a treatise on chords of circles. Some 300 years later, Ptolemy of Alexandria produced the *Almagest*, a large treatise on mathematical astronomy that ranks with Euclid's *Elements* and Appolonius' *Conics* as one of the great masterpieces of ancient times. One chapter of the *Almagest* consists of a table of chords which, in modern terminology, would be equivalent to a table of sines. The word *trigonometry*, from the Greek for triangle measurement, first appeared in print in the late 1590s. It superseded *goniometry*, the Greek word for angle measurement.

In trigonometry sines and cosines of acute angles appear as ratios of lengths of sides of a right triangle. Figure 23 shows a right triangle OAP with a right angle at vertex A. Let θ denote the measure of the angle at vertex O, and let a, b and c denote the lengths of the sides as shown in Figure 23, with

$$a = AP, \quad b = OA, \quad \text{and} \quad c = OP.$$

The numbers a and b are the lengths of the legs, and c is the length of the hypotenuse.

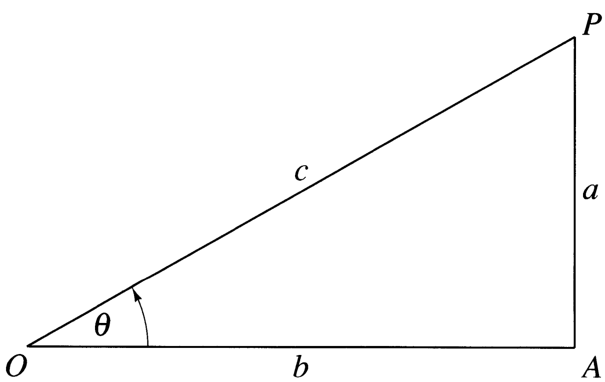


Figure 23. A right triangle with legs of length a and b and hypotenuse of length c.

The three ratios a/c, b/c, a/b and their reciprocals c/a, c/b, b/a play a fundamental role in trigonometry. When the triangle is magnified or contracted by a scaling factor k, each of the numbers a, b, c is multiplied by the same factor k, but the six ratios do not change, nor does the angle θ. Therefore the six ratios are completely determined by the shape of the triangle rather than by its size, which means they are determined by the angle θ.

The six ratios are called *trigonometric functions* of θ and they are given special names: sine, cosine, tangent, cosecant, secant, and cotangent. These functions are written more simply as sin, cos, tan, sec, csc, and cot, and are defined as follows:

$$\sin\theta = \frac{a}{c}, \quad \cos\theta = \frac{b}{c}, \quad \tan\theta = \frac{a}{b},$$

and their reciprocals

$$\csc\theta = \frac{c}{a}, \quad \sec\theta = \frac{c}{b}, \quad \cot\theta = \frac{b}{a}.$$

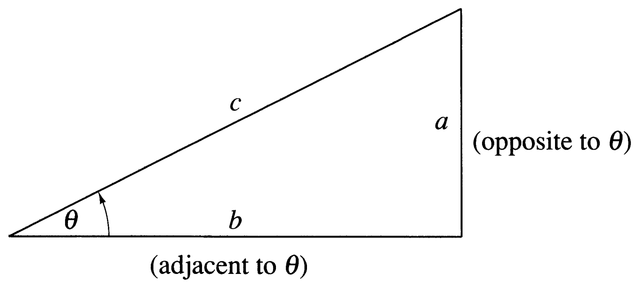


Figure 24. The six trigonometric functions are ratios of lengths of sides in a right triangle.

In Figure 24 the leg of length b is *adjacent* to the angle θ, and the leg of length a is *opposite* to θ. Therefore, the ratios defining the sine and cosine are sometimes written as follows:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}.$$

For acute angles the definitions of sine and cosine as ratios agree with the definitions given earlier as circular functions. To see this, we simply expand or contract the right triangle until its hypotenuse has length 1, as shown in Figure 25. Let x and y denote the lengths of the horizontal and vertical legs. From the earlier definition of the circular functions we have $x = \cos\theta$ and $y = \sin\theta$. To show that this definition of the sine and cosine agrees with the definition as ratios of sides of triangles we must verify that $x = b/c$ and $y = a/c$. But the two right triangles in Figure 25 are similar, so ratios of lengths of corresponding sides are equal. In particular, $x/1 = b/c$ and $y/1 = a/c$, which gives $x = b/c$ and $y = a/c$, as required.

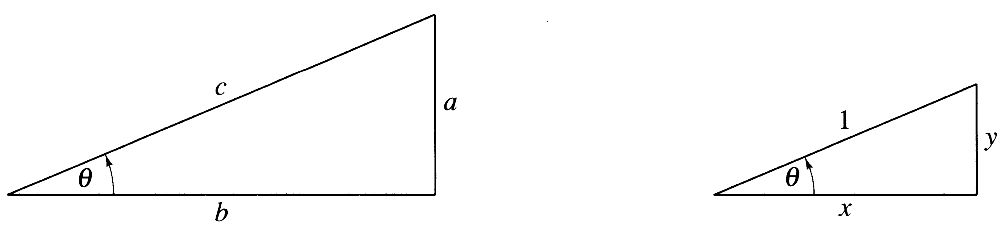


Figure 25. Ratios of lengths of corresponding sides of similar triangles are equal.

Although there are six trigonometric functions, only the sine and cosine need to be studied in detail, because the others can be expressed in terms of these two. In fact, we have

$$\tan\theta = \frac{\sin\theta}{\cos\theta}\,, \quad \sec\theta = \frac{1}{\cos\theta}\,, \quad \csc\theta = \frac{1}{\sin\theta}\,, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}\,,$$

when the denominators are not zero. We have already seen that the sine and cosine are related by the equations

$$\cos\theta = \sin(\tfrac{\pi}{2} - \theta) \quad \text{and} \quad \sin\theta = \cos(\tfrac{\pi}{2} - \theta).$$

There is also a relation connecting the sine and cosine of the same angle. This comes from the Theorem of Pythagoras for right triangles which states that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse. In the notation of Figure 25 the Pythagorean theorem states that

$$a^2 + b^2 = c^2.$$

When this equation is divided by c^2 it becomes $(a/c)^2 + (b/c)^2 = 1$. Replacing the ratios a/c and b/c by $\sin\theta$ and $\cos\theta$, we obtain the relation $(\sin\theta)^2 + (\cos\theta)^2 = 1$. This is usually written without the use of parentheses as follows,

$$\sin^2\theta + \cos^2\theta = 1,$$

and is sometimes referred to as the Pythagorean identity. The equation is valid for every real number θ.

Exercises on sines and cosines as ratios

1. (a) Use the Pythagorean theorem to show that in an isosceles right triangle the length of the hypotenuse is $\sqrt{2}$ times the length of each leg, as indicated in the diagram below.

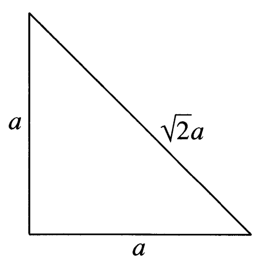
(b) From part (a), deduce the value of $\sin 45^\circ$ and of $\cos 45^\circ$.

2. When an equilateral triangle is bisected into two right triangles as shown in the diagram above, each is called a 30°-60° right triangle. Show that every 30°-60° right triangle is similar to one with legs 1 and $\sqrt{3}$, and hypotenuse of length 2.

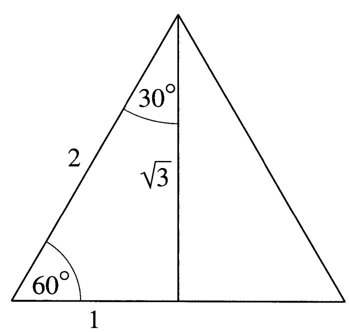
3. (a) Verify that $\sin 30^\circ = \frac{1}{2}$ and $\cos 30^\circ = \frac{1}{2}\sqrt{3}$.

(b) Calculate $\sin 60^\circ$ and $\cos 60^\circ$.

(c) Calculate $\tan 30^\circ$, $\tan 45^\circ$, and $\tan 60^\circ$.



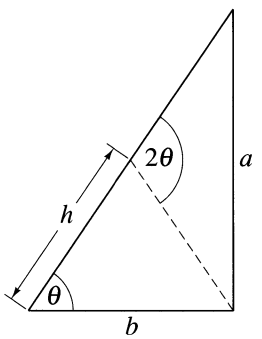
Exercise 1.



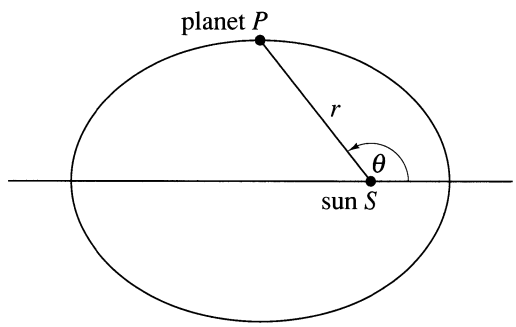
Exercise 2.

4. A ladder just reaches the top of a wall and forms a right triangle with base angle θ, base b amd altitude a, as shown in the diagram below. At a distance h feet up along the ladder a line from the right angle makes an angle 2θ with the ladder, as shown. Calculate a and b in terms of h, $\sin\theta$, and $\cos\theta$.

5. Cosines occur in the description of planetary motion. For example, a planet P moves around the sun S in such a way that its distance r from S satisifes an equation of the form $ed = r(1 + e\cos\theta)$, where d and e are constants (depending on the planet) with $0 < e < 1$, and θ is the angle that the line SP makes with a fixed axis through the sun as shown. (The orbit is called an *ellipse* with the sun at one focus, and e is called the eccentricity of the ellipse. For planet Earth, e is approximately 0.017 and the orbit is very nearly circular.) Show that the ratio of the nearest to the furthest distances of the planet from the sun is $(1 - e)/(1 + e)$.



Exercise 4.



Exercise 5.

6. The angles of a triangle are 3θ, 4θ, 5θ; the smallest side has length a. Find the lengths of all the sides in terms of a.

7. A ladder of length L is placed against a wall so that the angle it makes with the ground is twice the angle it makes with the wall. How far is the foot of the ladder from the wall?

8. On a map, town B lies midway between towns A and C; and towns B, C, and D are equidistant from each other. If the distance from A to C is 10 miles, find the distance from A to D.

9. A ladder 20 ft long just reaches the top of a wall when its foot is 13 ft from the base of the wall. When its foot is x feet from the wall the ladder makes an angle θ with the ground and projects 4 feet beyond the wall. Determine x and $\cos\theta$.

10. A person observes that the angular elevation of a column from the ground is 45°. After approaching 10 feet he finds the angular elevation is 60°. Determine the height of the column.

11. Three positions in the same horizontal plane are distant from each other 60, 80, 100 ft, respectively. An observer at each position finds that the angle of inclination to the top of a tower is 45°. Determine the height of the tower.

12. Find all values of θ such that: (a) $\sin\theta = \frac{1}{2}\tan\theta$. (b) $\sin^2\theta - \cos^2\theta = 1$.

13. Find all values of θ such that $\sin(x + \theta) = \cos(x - \theta)$ for every x.

Basic properties of the sine and cosine

In addition to the relations mentioned above there are a number of basic formulas that are satisfied by the sine and cosine and are frequently used in applications. The most important of these are listed here for easy reference. Properties (b) through (j) are called trigonometric identities. They are valid for all values of x and y.

Further discussion showing how these formulas are obtained will be given in a sequel to this program, *Sines and Cosines II.*

(a) **Special values:**

$$\sin 0 = 0, \qquad \sin \frac{\pi}{2} = 1, \qquad \sin \pi = 0, \qquad \sin \frac{3\pi}{2} = -1.$$

$$\cos 0 = 1, \qquad \cos \frac{\pi}{2} = 0, \qquad \cos \pi = -1, \qquad \cos \frac{3\pi}{2} = 0.$$

(b) **Periodicity:**

$$\sin(x + 2\pi) = \sin x \quad \text{and} \quad \cos(x + 2\pi) = \cos x \; .$$

(c) **The Pythagorean identity:**

$$\sin^2 x + \cos^2 x = 1 \; .$$

(d) **Odd property of the sine:**

$$\sin(-x) = - \sin x \; .$$

(e) **Even property of the cosine:**

$$\cos(-x) = \cos x \; .$$

(f) **Co-relations:**

$$\sin(\tfrac{\pi}{2} - x) = \cos x \quad \text{and} \quad \cos(\tfrac{\pi}{2} - x) = \sin x \; .$$

(g) **Addition formulas:**

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \; .$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y \; .$$

(h) **Subtraction formulas:**

$$\sin(x - y) = \sin x \cos y - \cos x \sin y \; .$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y \; .$$

(i) **Duplication formulas:**

$$\sin 2x = 2 \sin x \cos x \; .$$

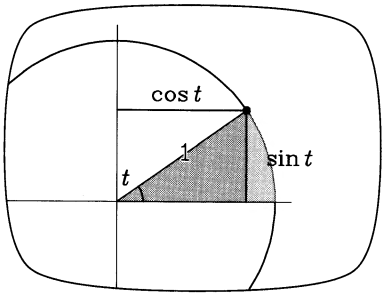
$$\cos 2x = \cos^2 x - \sin^2 x \; .$$

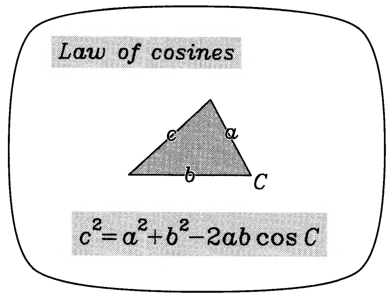
(j) **Difference formulas:**

$$\sin x - \sin y = 2 \sin \frac{x - y}{2} \cos \frac{x + y}{2} \; .$$

$$\cos x - \cos y = -2 \sin \frac{x - y}{2} \sin \frac{x + y}{2} \; .$$

6. *Recap of Part I, Preview of Part II*





In this program we've seen how sines and cosines emerge in many different ways. They occur as the vertical and horizontal coordinates of a point moving on a unit circle. They can be used to analyze graphs related to musical sounds or other periodic phenomena. And, they appear as ratios of lengths of sides of a right triangle. Because of this last property, sines and cosines are the building blocks of trigonometry, the study of triangles.

Hipparchus is considered the father of trigonometry because of his twelve books on properties of chords of circles. These books disappeared but many of the ideas survived in Ptolemy's *Almagest*, the oldest extant work containing properties of sines and cosines.

Sines and cosines are important, not only in geometry and trigonometry because of their relation to chords of circles or sides of triangles, but also because of the fundamental role they play in analyzing periodic oscillating systems. Mathematical properties of sines and cosines help us better understand oscillating systems such as a marble rolling in the bottom of a bowl, a mass oscillating at the end of a spring, a swinging pendulum, a vibrating guitar string, electrical impulses, radio waves, sound waves, and the wave theory of light. Sines and cosines also occur in the description of planetary motion.

In the next program, *Sines and Cosines II,* we'll look more closely at the fundamental properties of sines and cosines. For example, we'll discuss the law of cosines, an extension of the Pythagorean Theorem, which relates the lengths of the sides in any triangle. We'll also discuss the law of sines, which states that in any triangle the ratio of the length of a side to the sine of the opposite angle is constant. And, we'll derive formulas for finding the sine and cosine of the sum of two angles. (These are called addition formulas.) We'll also learn that if you combine a sine wave with a cosine wave of the same frequency, you just get another sine wave, possibly shifted.

When sines and cosines are calculated as ratios of sides of triangles, the accuracy of the final result may be limited by the degree of precision in measuring the lengths of sides. In *Sines and Cosines II* we will learn that the values of the sine and cosine functions can be calculated with great accuracy without using geometric diagrams. This is done by approximating these functions with polynomials. When you press a special function key on a hand calculator to find the value of sin x or cos x, the program inside the calculator often computes a polynomial approximation that gives the accuracy required.

Everything we hear or see comes to us in the form of sound waves or light waves. Because these are combinations of sine and cosine waves, sines and cosines are part of our everyday experience.

INDEX

Corrections

In the transition from word processor to printed page a tab stop error resulted in misalignment of a few lines of text on pages 14, 15 and 30. On page 14 the tables in Exercises 1 and 3 should appear as follows:

Exercise 1.

t	$\pi/4$	$3\pi/4$	$5\pi/4$	$7\pi/4$
$\sin t$	$\sqrt{2}/2$	$\sqrt{2}/2$	$-\sqrt{2}/2$	$-\sqrt{2}/2$

Exercise 3.

t	$\pi/6$	$\pi/3$	$2\pi/3$	$5\pi/6$	$7\pi/6$	$4\pi/3$	$5\pi/3$	$11\pi/6$
$\cos t$	$\sqrt{3}/2$	$1/2$	$-1/2$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$-1/2$	$1/2$	$\sqrt{3}/2$

In Figure 17(b) on page 21 the curve should pass through the origin.

Project MATHEMATICS!

COMPUTER ANIMATED MATHEMATICS VIDEOTAPES

Sines and Cosines I is part of a series of modules that use computer animation to help instructors teach basic concepts in mathematics. Each module consists of a videotape, about 20 minutes in length, and a workbook to guide the students through the video, elaborating on the important ideas. The modules are used as support material for existing courses in high school and community college classrooms, and may be copied without charge for educational use.

Based at the California Institute of Technology, *Project MATHEMATICS!* has attracted as partners the departments of education of 35 states in a consortium whose members reproduce and distribute the videotapes and written materials to public schools. The project is headed by Tom M. Apostol, professor of mathematics at Caltech and an internationally known author of mathematics textbooks. Co-director of the project is James F. Blinn, one of the world's leading computer animators, who is well known for his Voyager planetary flyby simulations. Blinn and Apostol worked together previously as members of the academic team that produced *The Mechanical Universe*, an award-winning physics course for television.

The first four videotapes produced by *Project MATHEMATICS!* have been distributed to thousands of classrooms nationwide. *The Theorem of Pythagoras* has received first-place awards at many international competitions, including a Gold Cindy at the 1989 Cindy Competition, Los Angeles, California. *The Story of Pi* was awarded a Gold Apple Award at the 1990 National Educational Film & Video Festival in Oakland, and a Red Ribbon Award at the 1990 American Film and Video Festival. *Similarity* was awarded a Silver Apple at the 1991 National Educational Film and Video Festival in Oakland, California.

Information about the project can be obtained by writing to the project director at the address on the title page of this booklet. Copies of the videotape and workbook on *The Theorem of Pythagoras, The Story of Pi, Similarity, Polynomials, The Teachers Workshop*, or *Sines and Cosines I* can be obtained at nominal charge from the Caltech Bookstore, 1-51 Caltech, Pasadena, CA 91125. (Tel: 818-356-6161) These titles can also be obtained from the National Council of Teachers of Mathematics, and the Mathematical Association of America at the addresses given below.

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